Mahatma Fule Arts, Comm. & Sitaramji Chaudhari
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DEPARTMENT OF MATHEMATICS

ADVANCED CALCULUS

Maxima and Minima of functions of two variables

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CLASSICAL MECHANICS

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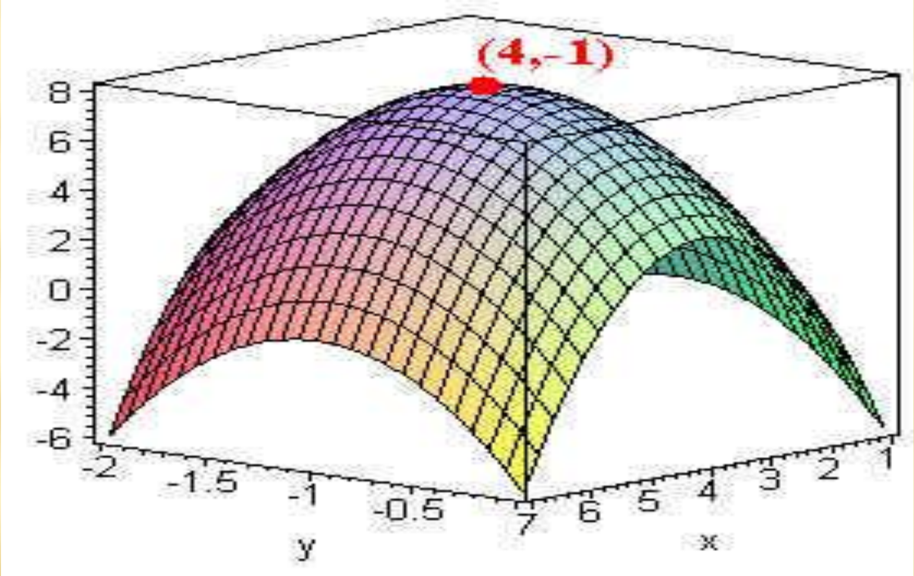
Local Maximum

$A\ function\ f(x,y)\ is\ said\ to\ be\ local\ \max.or\ relative\ \max.at\ (x_0,y_0)\ if$

$$f(x,y) \le f(x_0,y_0) \quad \forall\ (x,y)\ nbd\ of\ (x_0,y_0) \in D$$

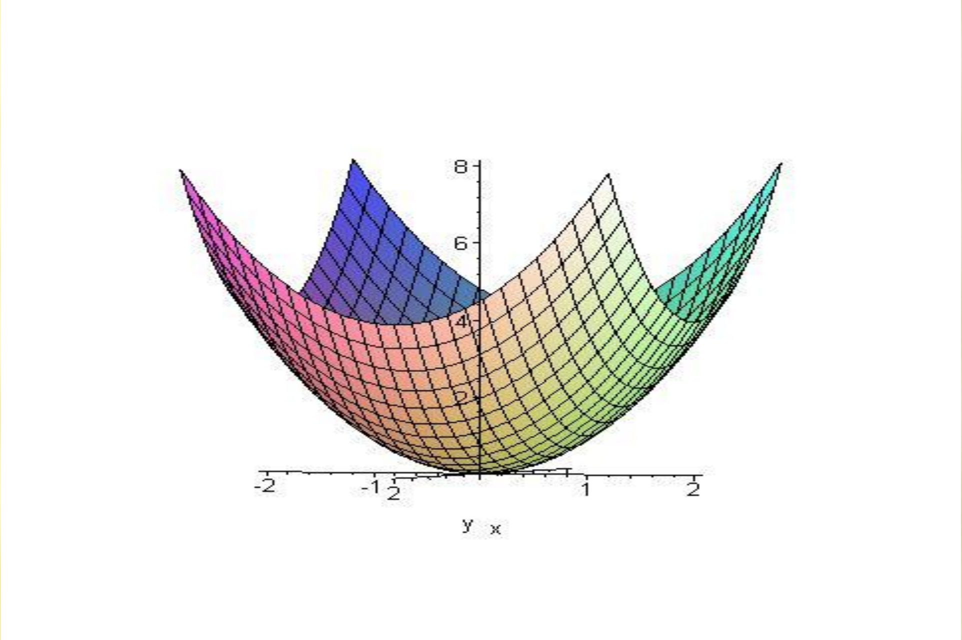


Local Minimum

$A\ function\ f(x,y)\ is\ said\ to\ be\ local\ \min.or\ relative$

$\min.at\ (x_0, y_0)\ if$

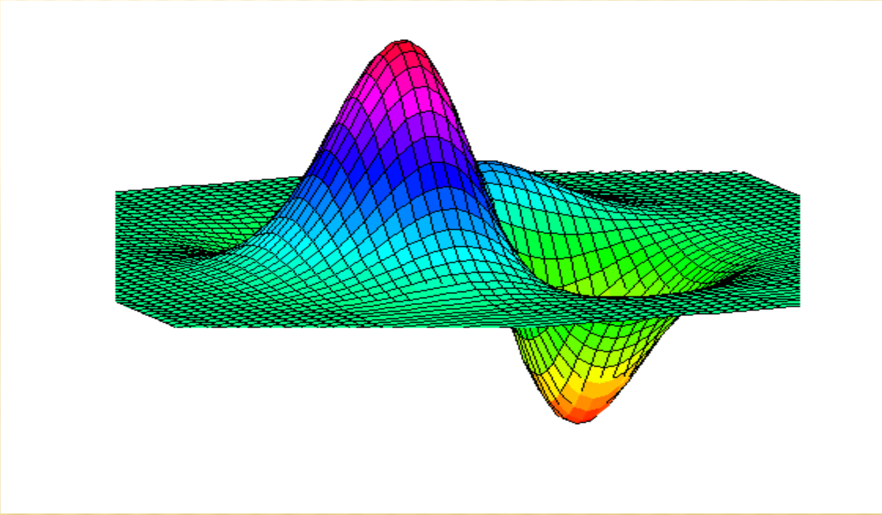
$$f(x,y) \geq f(x_0, y_0) \quad \forall\ (x,y) \in nbd\ of\ (x_0, y_0) \in D$$



Extrema

$$The\ local\ \max.\ or\ \min.\ value\ of\ f(x, y)$$

$$are\ called\ as\ extrema\ or\ extremum\ of\ f.$$



Necessary condition for maxima and minima

$Theorem:$

$Let\ f(x,y)\ be\ defined\ in\ an\ open\ region\ D\ and\ it\ has\ local\ \max.$
$or\ local\ \min.at\ (x_0,y_0).If\ the\ partial\ derivatives\ f_x and\ f_y$
$exists\ at\ (x_0,y_0), then\ f_x(x_0,y_0)=0\ and\ f_y(x_0,y_0)=0$

$\Pr oof:$

$Let\ f(x,y)\ be\ defined\ in\ an\ open\ region\ D\ and\ it\ has\ local\ \max.$
$at\ (x_0,y_0)$

$\therefore\ by\ definition\ of\ local\ \max.$

$$f(x,y) \le f(x_0,y_0) \qquad \forall\ (x,y) \in nbd\ of\ (x_0,y_0) \in D$$

$$Let\ \ y\ be\ fixed\ at\ y_0 \qquad (y = y_0)$$

$$\therefore \qquad f(x,y_0) \le f(x_0,y_0)$$

$$i.e. \qquad f(x_0 \pm h, y_0) \le f(x_0,y_0)$$

$$i.e. \qquad f(x_0 + h, y_0) \le f(x_0,y_0)\ and$$

$$f(x_0 - h, y_0) \le f(x_0,y_0)$$

$$f(x_0 + h, y_0) - f(x_0,y_0) \le 0\ and\ \ f(x_0 - h, y_0) - f(x_0,y_0) \le 0$$

$$\therefore \qquad \lim_{h\to 0^+} \frac{f(x_0 + h, y_{0)}) - f(x_0,y_0)}{h} \le 0 \quad and$$

$$\lim_{h\to 0^-} \frac{f(x_0 - h, y_{0)}) - f(x_0,y_0)}{-h} \ge 0$$

$$f_x(x_0, y_0) \le 0 \quad and \quad f_x(x_0, y_0) \ge 0 \Rightarrow f_x(x_0, y_0) = 0$$

$$Similarly,\ f_y(x_0, y_0) = 0$$

$$\therefore For\ local\ \max .of\ f(x, y)\ at\ (x_0, y_0) \in D,$$

$$f(x_0, y_0) = 0\ and\ f_y(x_0, y_0) = 0$$

$$Similarly\ \ if\ f\ has\ local\ \min .\ at\ (x_0, y_0)$$

$$then\ f_x(x_0, y_0) = 0\ and\ f_y(x_0, y_0) = 0$$

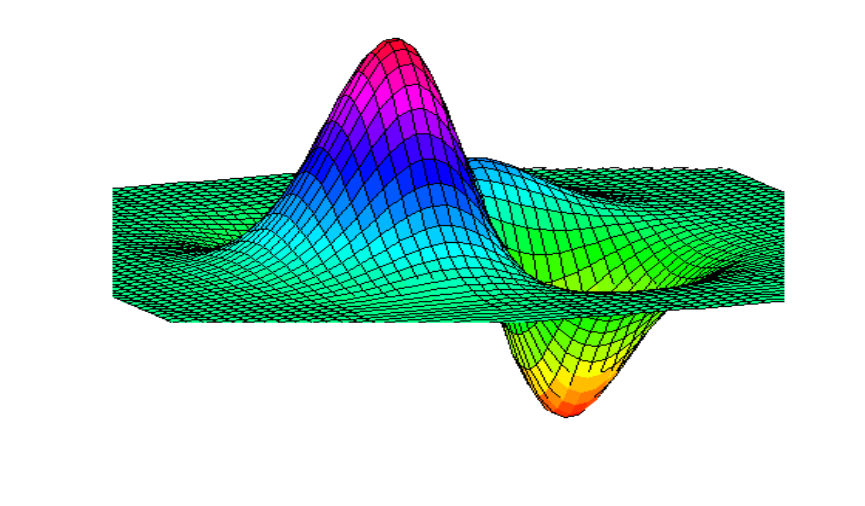
$$Hence\ theorem\ is\ proved. //$$

Critical point or stationary point:

A point at which $f_x = 0$ and $f_y = 0$ is called as critical point

i.e. a point $(x_0, y_0) \in D$ is said to be critical point

if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$



Saddle point:

A point at which the function have neither maximum or minimum is called as saddle point.

Second derivative test for maximum or minimum

Let f be a function of two variables defined in an open region $D \subseteq R^2$ and have continuous second order partial derivatives in D. Let (x_0, y_0) be a critical point of $f(x, y)$ and

$$r = f_{xx}(x_0, y_0)$$

$$s = f_{xy}(x_0, y_0)$$

$$t = f_{yy}(x_0, y_0)$$

$then$

$i)\ \ S^2 - rt \prec 0\ and\ \ r \prec 0$

$\Rightarrow\ f(x,y)\, has\, local\, \max .at\, (x_0, y_0)$

$ii)\ \ S^2 - rt \prec 0\ and\ \ r \succ 0$

$\Rightarrow\ f(x,y)\, has\, local\, \min .at\, (x_0, y_0)$

$iii)\ \ S^2 - rt \succ 0$

$\Rightarrow\ f(x,y)\ \ has\, neither\, \max .nor\, \min .at\, (x_0, y_0)$

$iv)\ \ S^2 - rt = 0$

$\Rightarrow\ test\ is\ inconclusive$

Example 1.

$$\textit{Find the } \min.\ \textit{value of} \quad u = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

Solution:

$$Let \qquad u = f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y} \qquad \ldots 1$$

$$\therefore \qquad \frac{\partial f}{\partial x} = y + a^3(-1/x^2) = y - \frac{a^3}{x^2} \quad and$$

$$\therefore \qquad \frac{\partial f}{\partial y} = x + a^3(-1/y^2) = x - \frac{a^3}{y^2}$$

For max. and min. (i.e. for critical point or stationary point)

$$\frac{\partial f}{\partial x} = 0 \quad and \quad \frac{\partial f}{\partial y} = 0$$

i.e. $\quad y - \frac{a^3}{x^2} = 0 \quad and \quad x - \frac{a^3}{y^2} = 0$

$\Rightarrow \quad x^2 y = a^3 \quad and \quad y^2 x = a^3$

$\Rightarrow \quad x^2 y = y^2 x \quad \Rightarrow \quad x^2 y - y^2 x = 0$

$\Rightarrow \quad xy(x - y) = 0 \quad \Rightarrow \quad x = y$

$\therefore \quad x^2 y = yx^2 = a^3 \quad \Rightarrow \quad x^3 = y^3 = a^3$

$\Rightarrow \quad x = y = a$ i.e. (a, a) is the critical point.

$$now \qquad r = \frac{\partial^2 f}{\partial x^2} = -a^3\left(-\frac{2}{x^3}\right) = 2\frac{a^3}{x^3} \quad , \qquad s = \frac{\partial^2 f}{\partial x \partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = -a^3\left(-\frac{2}{y^3}\right) = 2\frac{a^3}{y^3}$$

$$at\, the\, critical\ \ po\mathrm{int}\ \ (a, a)$$

$$r = \frac{\partial^2 f}{\partial x^2} = -a^3\left(-\frac{2}{x^3}\right) = 2\frac{a^3}{x^3} = 2\frac{a^3}{a^3} = 2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = -a^3\left(-\frac{2}{y^3}\right) = 2\frac{a^3}{y^3} = 2\frac{a^3}{a^3} = 2$$

$$\therefore\ s^2 - rt = 1^2 - 2.2 = -3 \prec 0 \ \ and\ \ r \succ 0$$

$$\therefore\ u = f(x, y)\, has \min .at\,(a, a)$$

$$\therefore\ \min .value\, of\ u\, is\ given\, by$$

$$u = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

$$u = aa + \frac{a^3}{a} + \frac{a^3}{a}$$

$$u = a^2 + a^2 + a^2 = 3a^2$$

$$\therefore \qquad u_{\min} = 3a^3. \ \ //$$

Example 2.

A circular Metallic plate has the shape of the region $x^2 + y^2 \le 1$

The entire plate is heated so that the temperature at any point (x, y)

is given by $T = 2x^2 + 3y^2 - x.$

Find the hottest and coldest points of the plate.

Solution:

Let the temperature at any point (x, y) *is given by subject to the metallic plate* $x^2 + y^2 \le 1$

$$\therefore \quad \frac{\partial T}{\partial x} = 4x - 1 \quad and \quad \frac{\partial T}{\partial y} = 6y$$

$$for \max . and \min . \quad \frac{\partial T}{\partial x} = 0 \quad and \quad \frac{\partial T}{\partial y} = 0$$

$$i.e. \quad 4x-1=0 \ \ and \ \ 6y=0 \quad \Rightarrow \quad x=\frac{1}{4} \quad and \quad y=0$$

$$\therefore \quad (\frac{1}{4},0) \ is\, the\, critical \ \ point$$

$$now \quad r=\frac{\partial^2 f}{\partial x^2}=4$$

$$s=\frac{\partial^2 f}{\partial x \partial y}=0$$

$$t=\frac{\partial^2 f}{\partial y^2}=6 \quad at \ (\frac{1}{4},0)$$

$$r=4, \quad s=0, \quad t=6$$

$$\therefore \qquad s^2-rt=0-4.6=-24 \prec 0 \quad \& \quad r \succ 0$$

$$\therefore \quad At\,(\frac{1}{4},0), \quad r \succ 0 \ \ and \ \ s^2-rt \prec 0$$

$$\therefore \quad T\ is \min .at \ \ (\frac{1}{4},0). \ //$$

$now\ \min.\ temperature\ at\ (1/4,0)$

$$so\ \min.\ temperature\ \ T = 2x^2 + 3y^2 - x$$

$$= 2(\frac{1}{4})^2 + 3.(0) - (\frac{1}{4}) = -\frac{1}{8}$$

$$\therefore \qquad T_{\min.} = -\frac{1}{8}.\ //$$

$Since\ (\frac{1}{4},0)\ is\ inside\ the\ plate\ \ x^2 + y^2 \le 1. The\ \max.of\ T\ may\ be$

$occur\ on\ the\ boundary\ of\ \ x^2 + y^2 \le 1$

$Consider\ \ x^2 + y^2 = 1\ \ i.e.\ \ y^2 = 1 - x^2$

$$\therefore\ T = 2x^2 + 3y^2 - x \ \Rightarrow\ T = 2x^2 + 3(1 - x^2) - x$$

$$\Rightarrow T = -x^2 - x + 3$$

$(\because\ T\ \text{is a single variable function})$

$$\therefore\ \frac{\partial T}{\partial x} = -2x - 1$$

$$now\qquad y^2 = (1 - x^2) \Rightarrow y^2 = 1 - (-1/2)^2$$

$$\Rightarrow y^2 = 1 - 1/4 = 3/4$$

$$\Rightarrow y = \pm\frac{\sqrt{3}}{2}$$

$$and\qquad \frac{\partial^2 T}{\partial x^2} = -2$$

$$\therefore\ T\ \text{is max, at}\ \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\ \text{and}\ \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

So max .*temperature*

$$T = 2x^2 + 3y^2 - x.$$

$$= 2(-\frac{1}{2})^2 + 3(\pm\frac{\sqrt{3}}{2})^2 - (-\frac{1}{2})$$

$$= 2.\frac{1}{4} + 3.\frac{3}{4} + \frac{1}{2} = \frac{13}{4}$$

$$\therefore \quad T_{\max.} = \frac{13}{4}$$

now min. *temperature at* (1/4,0)

so min. *temperature*

$$T = 2x^2 + 3y^2 - x$$

$$= 2(\frac{1}{4})^2 + 3.(0) - (\frac{1}{4}) = -\frac{1}{8}$$

$$\therefore \quad T_{\min.} = -\frac{1}{8}. \quad //$$

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Thanks…….!